

Bounding the first Dirichlet eigenvalue of a tube around a complex submanifold of $\mathbb{C}P^n(\lambda)$ by the degrees of the polynomials defining it

M. Carmen Domingo-Juan and Vicente Miquel

Abstract

We obtain upper bounds for the first Dirichlet eigenvalue of a tube around a complex submanifold P of $\mathbb{C}P^n(\lambda)$ which depends only on the radius of the tube, the degrees of the polynomials defining P and the first eigenvalue of some model centers of the tube. The bounds are sharp on these models. Moreover, when the models used are $\mathbb{C}P^q(\lambda)$ or $Q^{n-1}(\lambda)$ these bounds also give gap phenomena and comparison results.

1 Introduction

This paper concerns with classical Dirichlet eigenvalue problem

$$\Delta f = \mu f \quad \text{on } M \quad \text{and} \quad f = 0 \quad \text{on } \partial M \quad (1)$$

on a connected compact Riemannian manifold M with boundary ∂M . An active area of research in this problem is to determine “interesting” upper or lower bounds for the first eigenvalue $\mu_1(M)$ corresponding to equation (1). Here “interesting” means sharp or related with some special properties of the space M . As examples of the work in this direction we have the results of S. Y. Cheng ([3]), R. Reilly ([17]), M. Gage ([6]), A. Kasue ([11]), J. M. Lee ([13]), F. Giménez, A. Lluch, V. Palmer and the second author ([7, 14, 15, 16]), G. P. Bessa and J. F. Montenegro ([2]). In these quoted papers, M is a tube around some compact submanifold of some Riemannian manifold or a manifold with boundary with special bounds on the curvatures of M and ∂M . The bounds obtained for $\mu_1(M)$ are usually the corresponding to some special model tube and, many times, the equality characterizes this model tube.

Very close to the the problem of obtaining bounds for $\mu_1(M)$ is the obtention of estimates for the volume of M . The deep relation between these problems is well known and some of the papers quoted before show or explore it. Concerning the volume, A. Gray (see [8, 9, 10]) has shown that the volume of a tube around a complex submanifold of $\mathbb{C}P^n(\lambda)$ can be stated in terms of the degrees of the polynomials defining the center of the tube. It is natural, then, to ask for some relation between the first Dirichlet eigenvalue of the tube and the degrees of the polynomials defining its center. Another hope for getting such kind of relation are the recent results by Colbois, Dryden and El Soufi ([4]) where they obtain bounds of the first eigenvalue for the closed problem on algebraic submanifolds in

the euclidean space in terms of the degrees of the polynomials defining them. The problem was addressed in [5], where A. Lluch and the authors obtained a bound of the first Dirichlet eigenvalue of a tube around a complex curve in the complex projective space in terms of the degrees of the polynomial defining the complex curve. Our aim in this paper is to complete the work started in [5] by studying tubes around complex submanifolds of any dimension and , also, giving bounds of different flavor obatinig the bound by comparison with the eigenvalues of tubes around all the possible homogeneous complex submanifolds for which the eigenfunction corresponding to $\mu_1(M)$ are radial.

We state now our results with more precision. Let $\mathbb{C}P^n(\lambda)$ be the complex projective space of holomorphic sectional curvature 4λ , and let $\wp : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n(\lambda)$ be the canonical projection. Chow's Theorem states that every complete complex submanifold P^q of $\mathbb{C}P^n(\lambda)$ (of complex dimension q) is the image by \wp of the set of zeroes of $n - q$ homogeneous polynomials of degrees a_{q+1}, \dots, a_n . For such a submanifold, we shall denote by P_ρ the tube of radius $\rho > 0$ around P and by ∂P_ρ^q its boundary. We shall always consider ρ lower than the cut distance $\text{cut}(P)$ from P^q . In [5] it is studied the Dirichlet eigenvalue problem (1) for $M = P_\rho^1$, for P a complex curve of $\mathbb{C}P^n(\lambda)$ and it is obtained an upper bound of $\mu_1(P_\rho)$ of the form

$$\mu_1(P_\rho^1) \leq \mu_1(\mathbb{C}P^1(\lambda)_\rho) - M^1(\rho, n, a_2, \dots, a_q), \quad (2)$$

where $\mathbb{C}P^1(\lambda)$ is embedded as a totally geodesic complex submanifold of $\mathbb{C}P^n(\lambda)$, $M^1(\rho, n, 1, \dots, 1) = 0$ and $M^1(\rho, n, a_2, \dots, a_q) > 0$ for $(a_2, \dots, a_n) \neq (1, \dots, 1)$. Moreover, the equality is attained if and only if $P^1 = \mathbb{C}P^1(\lambda)$ (that is, $(a_2, \dots, a_q) = (1, \dots, 1)$).

This bound address the problem of relating $\mu_1(P_r)$ with the degrees of the polynomials defining P . Moreover it is also a comparison theorem with $\mu_1(\mathbb{C}P^1(\lambda)_r)$ and shows a gap phenomon for $\mu_1(P_\rho^1)$ between the case $P^1 = \mathbb{C}P^1(\lambda)$ (which corresponds to $a_2 = \dots = a_n = 1$) and the other complex submanifolds P , and states that the gap is measured by the degrees a_s (a similar gap phenomenon occurs in the study of the closed eigenvalue problem for complex submanifolds by J. P. Bourguignon, P. Li and S. T. Yau ([1])).

In this paper we address the problem of getting bounds of the same nature that (2) for higher dimensions of P^q ($q \geq 1$) and comparing with tubes around some model complex submanifolds. These models, denoted by \mathfrak{P} , are the complex submanifolds of $\mathbb{C}P^n(\lambda)$ having constant normal curvatures (constant means here that it does not depend on the point nor on the direction). They were classified by Kimura ([12]) and are listed at the end of section 2. As in [5], we use the deep ideas in the work of A. Gray ([8], [9], [10]) to get theorems. But here we use them more form the root, which gives simpler computations and also more general results. Of course, another ingredient is the work of Kimura ([12]). We shall prove:

Theorem 1.1 *For $1 \leq q \leq n - 1$, $q = \dim_{\mathbb{C}} P = \dim_{\mathbb{C}} \mathfrak{P}$, the first eigenvalue $\mu_1(P_\rho^q)$ of the Dirichlet eigenvalue problem (1) satisfies the inequalities*

$$\mu_1(P_\rho^q) \leq \mu_1(\mathfrak{P}_\rho) + M_{\mathfrak{P}}(\rho, n, q, a_{q+1}, \dots, a_n), \quad (3)$$

where $M_{\mathfrak{P}}(\rho, n, q, a_{q+1}, \dots, a_n)$ are well defined constants which depends only on $\mathfrak{P}, \rho, n, q, a_{q+1}, \dots, a_n$. Moreover:

1. $M_{\mathbb{C}P^q(\lambda)} \leq 0$ and the equality holds if and only if $P = \mathbb{C}P^q(\lambda)$, which is equivalent to $\sum_{i=q+1}^n a_i = n - q$, then (3) also gives a gap between $\mu_1(\mathbb{C}P^n(\lambda)_\rho)$ and the corresponding eigenvalues of tubes with the same radius around complex submanifolds defined by polynomials of higher degree.
2. When $q = n - 1$ and $\mathfrak{P} = Q^{n-1}(\lambda)$ (the complex hyperquadric), there is a ρ_0 , $0 < \rho_0 \leq \text{cut}(P)$, depending on the degree a_n of the polynomial defining P , such that, for every $\rho < \rho_0$, $M_{Q^{n-1}(\lambda)} \leq 0$ if $P \neq \mathbb{C}P^n(\lambda)$ (which equivalent to $a_n > 1$) and, in this case, equality holds if and only if the polynomial defining P has degree 2. As a consequence, for every $\rho < \rho_0$:
 - (a) $Q^{n-1}(\lambda)$ gives the biggest first eigenvalue of problem (1) among all the tubes around a complex hypersurface defined by an homogeneous polynomial of degree 2, and
 - (b) For all the complex hypersurfaces defined by polynomials of degree ≥ 3 , there is a gap between $\mu_1(Q^{n-1}(\lambda)_\rho)$ and $\mu_1(P_\rho)$.

Remark 1.2 We think that $\rho_0 = \text{cut}(P)$, but we have not enough precise bounds for ρ_0 to assure it. For the other models \mathfrak{P} different from $\mathbb{C}P^n(\lambda)$ and $Q^{n-1}(\lambda)$ we have no control on the term $M_{\mathfrak{P}}$. This is what make us unable to obtain any kind of comparison theorem from the bound (μ_1) for these other models.

If \mathfrak{P} is not any of the models we have considered, the μ_1 -eigenfunction is not radial, then our method cannot give an upper bound of $\mu_1(P_\rho) - \mu_1(\mathfrak{P}_\rho)$ depending only on \mathfrak{P} , n , q , ρ and the degrees of the polynomials defining P .

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2 Preliminaries on complex submanifolds and the tubes around them

Given a complex submanifold P of $\mathbb{C}P^n(\lambda)$ of real dimension $2q$, we shall denote by r the distance to P in $\mathbb{C}P^n(\lambda)$. Let us denote by $\mathcal{N}P$ the normal bundle of P , by A_ξ the Weingarten map of P in the direction of $\xi \in \mathcal{N}P$, $|\xi| = 1$. Moreover, we shall use the notations s_λ and c_λ for

$$s_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, \quad c_\lambda(t) = \cos(\sqrt{\lambda}t), \quad \text{ta}_\lambda(t) = \frac{s_\lambda(t)}{c_\lambda(t)}$$

which satisfy the computational rules $s'_\lambda = c_\lambda$ and $c'_\lambda = -s_\lambda$.

Since P is a complex submanifold, given $\xi \in \mathcal{N}P$, $|\xi| = 1$, the Weingarten map A_ξ , has eigenvalues $k_1(\xi), -k_1(\xi), \dots, k_q(\xi), -k_q(\xi)$. The trace of the Weingarten map $S(t)$ of

∂P_t is (cf. [10], page 125, formula (7.25)):

$$\text{tr } S = 2s_\lambda(\rho)c_\lambda(\rho) h_P(\rho) - (2n - 2q - 1) \frac{c_\lambda(\rho)}{s_\lambda(\rho)} + \lambda \frac{s_\lambda(\rho)}{c_\lambda(\rho)}, \quad (4)$$

$$\text{where } h_P(\rho) = \sum_{i=1}^q \frac{(\lambda + k_i^2)}{c_\lambda^2(\rho) - k_i^2 s_\lambda^2(\rho)} \quad (5)$$

On the other hand, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^∞ function, one has (cf. [15] for instance):

$$\Delta(f \circ r) = -f'' \circ r + \text{tr } S f' \circ r \quad (6)$$

From now on we shall omit the writing of “ $\circ r$ ” when it can be understood by the context.

The volume element ω of a tube P_ρ in Fermi coordinates around P can be written (cf. [10], page 125 formula (7.26)) as

$$\omega = \theta(p, \xi, r) d\xi dp dr, \text{ with } \theta(p, \xi, r) = s_\lambda^{2n-2q-1} c_\lambda v(p, \xi, r), \quad (7)$$

$$\text{where } v(p, \xi, r) = \prod_{j=1}^q (c_\lambda^2 - s_\lambda^2 k_j(\xi)^2) = c_\lambda^{2q} \prod_{j=1}^q (1 - \text{ta}_\lambda^2 k_j(\xi)^2),$$

where dp and $d\xi$ denote, respectively, the volume elements of P and $S^{2n-2q-1}$. We remark that, for p and ξ fixed, the first positive value of r where $v(p, \xi, r)$ (then $\theta(p, \xi, r)$) vanishes is lower than $\text{cut}(P)$.

Developing the product or the determinant in the above formula, one obtains

$$\prod_{i=1}^q (1 - \text{ta}_\lambda^2 k_i(\xi)^2) = \sum_{c=0}^q \Psi_{2c}(\xi, \dots, \xi) \text{ta}_\lambda^{2c}, \text{ with} \quad (8)$$

$$\Psi_{2c}(\xi, \dots, \xi) = (-1)^c \sum_{\substack{i_1, \dots, i_c = 1 \\ i_1 < \dots < i_c}}^q k_{i_1}^2(\xi) \dots k_{i_c}^2(\xi) \quad (9)$$

And the ψ_{2c} satisfy (cf. [10], pages 65 and 125):

$$\int_{S^{2n-2q-1}} \Psi_{2c}(\xi, \dots, \xi) d\xi =: I_{2c}(\Psi_{2c}) = a(c) \mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c), \quad (10)$$

where

$$a(c) = \frac{2\pi^{n-q}}{c! (2c)! 2^c (n - q + c - 1)!} \quad (11)$$

and $\mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c)$ is a contraction of the curvature operator $(R^P - R^{\mathbb{C}P^n(\lambda)})^c$ which is related with the Chern form $\bar{\gamma}_c = \gamma_c(R^P - R^{\mathbb{C}P^n(\lambda)})$ of the curvature operator $(R^P - R^{\mathbb{C}P^n(\lambda)})$ by

$$b(c) := \int_P \mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c) dp = \frac{i! (2c)! (2\pi)^c}{(q - c)!} \int_P \bar{\gamma}_c \wedge F^{q-c} dp. \quad (12)$$

One can look at [10] page 56 for a precise definition of \mathcal{C}^{2c} and page 88 for the definition of γ_c . On the other hand, $\overline{\gamma}_c$ is related with the degrees a_j of the polynomials defining P (cf. [10] page 141) by

$$[\overline{\gamma}] = [\gamma(R^P - R^{\mathbb{C}P^n(\lambda)})] = \left[\frac{1}{\prod_{j=q+1}^n \left(1 + \frac{(a_j - 1)\lambda}{\pi} F \right)} \right], \quad (13)$$

where $[\cdot]$ denotes the cohomology class of the corresponding differential form and the $\overline{\gamma}_c$ are defined from $\overline{\gamma}$ by

$$\overline{\gamma} = 1 + \overline{\gamma}_1 + \cdots + \overline{\gamma}_q + \dots$$

From which it follows that

$$1 = (1 + \overline{\gamma}_1 + \cdots + \overline{\gamma}_q + \dots) \prod_{j=q+1}^n \left(1 + (a_j - 1) \frac{\lambda}{\pi} F \right),$$

then, for $c = 1, \dots, q$,

$$\overline{\gamma}_c = (-1)^c \beta_c \left(\frac{\lambda}{\pi} F \right)^c \quad \text{where} \quad \beta_c = \sum_{\substack{j_1, \dots, j_c = q+1 \\ j_1 \leq \dots \leq j_c}}^n (a_{j_1} - 1) \dots (a_{j_c} - 1) \quad (14)$$

By substitution of (14) in (12) one obtains

$$b(c) = (-1)^c \frac{c! (2c)! (2\pi)^c \lambda^c \beta_c}{(q-c)! \pi^c} \int_P F \wedge \dots \wedge F \, dp = (-1)^c \frac{c! (2c)! 2^c q!}{(q-c)!} \lambda^c \beta_c \, vol(P). \quad (15)$$

In [12], Kimura classified all the complex submanifolds of $\mathbb{C}P^n(\lambda)$ whose principal curvatures are constant in the sense that they depend neither on the point of the submanifold nor on the normal vector. They are:

- Totally geodesic $\mathbb{C}P^q(\lambda)$. It has $k_i = 0$ (16)

- The complex hyperquadric $Q^{n-1}(\lambda)$, where $k_i(\xi) = \sqrt{\lambda}$ for $i = 1, \dots, n-1$. (17)

- $\mathbb{C}P^1(\lambda) \times \mathbb{C}P^{m-1}(\lambda) \subset \mathbb{C}P^{2m-1}(\lambda)$ for $m \geq 3$ (then $n = 2m-1 \geq 5$ (18)

and $q = m$), where $k_1 = k_2 = \sqrt{\lambda}$, $k_3 = \dots = k_m = 0$

- $SU(5)/S(U(3) \times U(2)) \subset \mathbb{C}P^9(\lambda)$ (then $n = 9$, $q = 6$), (19)

where $k_1 = k_2 = k_3 = k_4 = \sqrt{\lambda}$, $k_5 = k_6 = 0$.

- $SO(10)/U(5) \subset \mathbb{C}P^{15}(\lambda)$ (then $n = 15$, $q = 10$), (20)

where $k_1 = \dots = k_6 = \sqrt{\lambda}$, $k_7 = \dots = k_{10} = 0$

3 Proof of Theorem 1.1

We shall denote by $S_{\mathfrak{P}}$ the Wingarten maps of the tubular hypersurfaces centred at a model comparison \mathfrak{P} and by $f_{\mathfrak{P}}$ an eigenfunction corresponding to $\mu_1(\mathfrak{P}_\rho)$. According to (6), $f_{\mathfrak{P}}$ is the solution of the equation

$$-f_{\mathfrak{P}}'' + \text{tr } S_{\mathfrak{P}} f_{\mathfrak{P}}' = \mu_1(\mathfrak{P}_\rho) f_{\mathfrak{P}}, \quad f_{\mathfrak{P}}(\rho) = 0, \quad f_{\mathfrak{P}}'(0) = 0. \quad (21)$$

This function satisfies the inequalities (cf. [14] or [15] for instance)

$$f_{\mathfrak{P}} > 0 \text{ on } [0, \rho[\quad \text{and} \quad f_{\mathfrak{P}}' < 0 \text{ on }]0, \rho]. \quad (22)$$

We shall apply the Raileigh's theorem using $f_{\mathfrak{P}} \circ r$ as a test function, then

$$\mu_1(P_\rho) \leq \frac{\int_{P_\rho} f_{\mathfrak{P}}(\Delta f_{\mathfrak{P}})}{\int_{P_\rho} f_{\mathfrak{P}}^2} \quad (23)$$

Let us compute the right hand side of the above inequality. From (4), (6) and (21), we get

$$\begin{aligned} \Delta(f_{\mathfrak{P}} \circ r) &= -f_{\mathfrak{P}}'' \circ r + \text{tr } S_{\mathfrak{P}} f_{\mathfrak{P}}' \circ r + (\text{tr } S - \text{tr } S_{\mathfrak{P}}) f_{\mathfrak{P}}' \circ r \\ &= \mu_1(\mathfrak{P}_\rho) f_{\mathfrak{P}} + 2s_\lambda \circ r \, c_\lambda \circ r (h_P - h_{\mathfrak{P}}) f_{\mathfrak{P}}' \end{aligned} \quad (24)$$

From (23), (24), (16) and (17) one gets

$$\begin{aligned} \mu_1(P_\rho) &\leq \mu_1(\mathfrak{P}_\rho) + \frac{\int_{P_\rho} f_{\mathfrak{P}} 2 s_\lambda c_\lambda (h_P - h_{\mathfrak{P}}) f_{\mathfrak{P}}' \omega}{\int_{P_\rho} f_{\mathfrak{P}}^2 \omega} \\ &= \mu_1(\mathfrak{P}_\rho) + \frac{\int_0^\rho \int_P \int_{S^{2n-2q-1}} 2 f_{\mathfrak{P}} f_{\mathfrak{P}}' s_\lambda^{2n-2q} c_\lambda^2 (h_P - h_{\mathfrak{P}}) v(p, \xi, r) d\xi dp dr}{\int_0^\rho \int_P \int_{S^{2n-2q-1}} f_{\mathfrak{P}}^2 \theta(p, \xi, r) d\xi dp dr}, \end{aligned} \quad (25)$$

where we have used the expression (7) of the volume element ω of P_ρ^q in Fermi coordinates.

Let us work with the integrand of the numerator in (25). Using (7), (8) and (9) we get

$$\begin{aligned} h_P v(p, \xi, r) &= \left(\sum_{i=1}^q \frac{k_i(\xi)^2 + \lambda}{c_\lambda^2 - k_i(\xi)^2 s_\lambda^2} \right) \prod_{j=1}^q (c_\lambda^2 - k_j(\xi)^2 s_\lambda^2) \\ &= \sum_{i=1}^q (k_i(\xi)^2 + \lambda) c_\lambda^{2q-2} \prod_{\substack{j=1 \\ j \neq i}}^q (1 - k_j(\xi)^2 \text{ta}_\lambda^2) \end{aligned} \quad (26)$$

$$\begin{aligned} h_{\mathfrak{P}} v(p, \xi, r) &= h_{\mathfrak{P}} \prod_{j=1}^q (c_\lambda^2 - k_j(\xi)^2 s_\lambda^2) = h_{\mathfrak{P}} c_\lambda^{2q} \sum_{c=0}^q (-1)^c \text{ta}_\lambda^{2c} \sum_{\substack{i_1, \dots, i_c = 1 \\ i_1 < \dots < i_c}}^q k_{i_1}^2 \dots k_{i_c}^2 \end{aligned} \quad (27)$$

But direct computation gives

$$\sum_{i=1}^q k_i(\xi)^2 \prod_{\substack{j=1 \\ j \neq i}}^q (1 - k_j(\xi)^2 \text{ta}_\lambda^2) = \sum_{c=1}^q (-1)^{c-1} c \text{ta}_\lambda^{2c-2} \sum_{\substack{i_1, \dots, i_c = 1 \\ i_1 < \dots < i_c}}^q k_{i_1}^2 \dots k_{i_c}^2 \quad (28)$$

and

$$\sum_{i=1}^q \lambda \prod_{\substack{j=1 \\ j \neq i}}^q (1 - k_j(\xi)^2 \text{ta}_\lambda^2) = \lambda \sum_{c=1}^q (-1)^{c-1} (q - c + 1) \text{ta}_\lambda^{2c-2} \sum_{\substack{i_2, \dots, i_c = 1 \\ i_2 < \dots < i_c}}^q k_{i_2}^2 \dots k_{i_c}^2 \quad (29)$$

From (26), (28), (29), (9), (10) and (12) it follows

$$\begin{aligned} & \int_P \int_{S^{2n-2q-1}} h_P v(p, \xi, r) \\ &= c_\lambda^{2q-2} \left(- \sum_{i=1}^q i \text{ta}_\lambda^{2i-2} a(i) b(i) + \lambda \sum_{i=1}^q (q - i + 1) \text{ta}_\lambda^{2i-2} a(i-1) b(i-1) \right) \end{aligned} \quad (30)$$

and

$$\int_P \int_{S^{2n-2q-1}} h_{\mathfrak{P}} v(p, \xi, r) = h_{\mathfrak{P}} c_\lambda^{2q} \sum_{i=0}^q \text{ta}_\lambda^{2i} a(i) b(i) \quad (31)$$

Now, after the substitution of (30) and (31) in the numerator of (25), having into account that $2f_{\mathfrak{P}} f'_{\mathfrak{P}} = (f_{\mathfrak{P}}^2)'$, we compute for that numerator:

$$\begin{aligned} & \int_0^\rho \int_P \int_{S^{2n-2q-1}} 2f_{\mathfrak{P}} f'_{\mathfrak{P}} s_\lambda^{2n-2q} c_\lambda^2 (h_P - h_{\mathfrak{P}}) v(p, \xi, r) d\xi dp dr \\ &= \int_0^\rho (f_{\mathfrak{P}}^2)' s_\lambda^{2n-2q} c_\lambda^{2q} \left(- \sum_{i=1}^q i \text{ta}_\lambda^{2i-2} a(i) b(i) + \lambda \sum_{i=1}^q (q - i + 1) \text{ta}_\lambda^{2i-2} a(i-1) b(i-1) \right. \\ & \quad \left. - h_{\mathfrak{P}} c_\lambda^2 \sum_{i=0}^q \text{ta}_\lambda^{2i} a(i) b(i) \right) dr \\ &= \int_0^\rho (f_{\mathfrak{P}}^2)' s_\lambda^{2n-2q} c_\lambda^{2q} \sum_{i=0}^q (-i \text{ta}_\lambda^{2i-2} + \lambda (q - i) \text{ta}_\lambda^{2i} - h_{\mathfrak{P}} c_\lambda^2 \text{ta}_\lambda^{2i}) a(i) b(i) dr =: \mathcal{N} \end{aligned} \quad (32)$$

But in all our comparison models \mathfrak{P} the possible values for k_i are $\pm\sqrt{\lambda}$ or 0. Let \mathfrak{z} be the number of normal curvatures with value $\sqrt{\lambda}$. With this conventon, it follows from (5) that the expression for $h_{\mathfrak{P}} c_\lambda^2$ is

$$h_{\mathfrak{P}} c_\lambda^2 = \mathfrak{z} \lambda \frac{1}{c_{4\lambda}} + q\lambda \quad (33)$$

We continue with the above computation (32)

$$\begin{aligned}
\mathcal{N} &= \int_0^\rho (f_{\mathfrak{P}}^2)' s_\lambda^{2n-2q} c_\lambda^{2q} \sum_{i=0}^q \left(-i \operatorname{ta}_\lambda^{2i-2} c_\lambda^{-2} + \lambda q \operatorname{ta}_\lambda^{2i} - \left(\mathfrak{z} \lambda \frac{1}{c_{4\lambda}} + q\lambda \right) \operatorname{ta}_\lambda^{2i} \right) a(i) b(i) dr \\
&= \sum_{i=0}^q \int_0^\rho (f_{\mathfrak{P}}^2)' \left(-i - \mathfrak{z} \lambda \frac{s_\lambda^2}{c_{4\lambda}} \right) s_\lambda^{2(n-q+i-1)} c_\lambda^{2(q-i)} a(i) b(i) dr \\
&= \sum_{i=0}^q \int_0^\rho (f_{\mathfrak{P}}^2)' \left(\frac{-i c_\lambda^2 + (-\mathfrak{z} + i) \lambda s_\lambda^2}{c_{4\lambda}} \right) s_\lambda^{2(n-q+i-1)} c_\lambda^{2(q-i)} a(i) b(i) dr \\
&= \int_0^\rho (f_{\mathfrak{P}}^2)' \frac{1}{c_{4\lambda}} s_\lambda^{2(n-q-1)} c_\lambda^{2q} \sum_{i=0}^q \left(-i c_\lambda^2 + (i - \mathfrak{z}) \lambda s_\lambda^2 \right) \operatorname{ta}_\lambda^{2i} a(i) b(i) dr \tag{34}
\end{aligned}$$

On the other hand, for the denominator of (25), from (7), the remark after (7), (8), (10) and (12) it follows:

$$\begin{aligned}
0 &< \int_0^\rho \int_P \int_{S^{2n-2q-1}} f_{\mathfrak{P}}^2 \theta(p, \xi, r) d\xi dp dr = \int_0^\rho f_{\mathfrak{P}}^2 s_\lambda^{2n-2q-1} c_\lambda^{2q+1} \sum_{i=0}^q \operatorname{ta}_\lambda^{2i} a(i) b(i) dr \\
&= \sum_{i=0}^q a(i) b(i) \int_0^\rho f_{\mathfrak{P}}^2 s_\lambda^{2n-2q+2i-1} c_\lambda^{2q-2i+1} dr \tag{35}
\end{aligned}$$

Now we shall use the notation

$$\mu_{\mathfrak{P}} = f_{\mathfrak{P}}^2 s_\lambda^{2n-2q-1} c_\lambda^{2q+1} \quad \text{and} \quad \nu_{\mathfrak{P}} = (f_{\mathfrak{P}}^2)' \frac{s_\lambda^{2n-2q-1} c_\lambda^{2q}}{c_{4\lambda}}. \tag{36}$$

Substitution of (34) and (35) in (25) gives

$$\begin{aligned}
\mu_1(P_\rho) &\leq \mu_1(\mathfrak{P}_\rho) \\
&+ \frac{\int_0^\rho \sum_{i=0}^q \left(-i c_\lambda^2 + (i - \mathfrak{z}) \lambda s_\lambda^2 \right) \operatorname{ta}_\lambda^{2i} a(i) b(i) \nu_{\mathfrak{P}} dr}{\sum_{i=0}^q \int_0^\rho s_\lambda^{2i} c_\lambda^{-2i} a(i) b(i) \mu_{\mathfrak{P}} dr}, \tag{37}
\end{aligned}$$

From the formulae (11) and (15) for $a(i)$ and $b(i)$ it follows that

$$\frac{a(i) b(i)}{a(0) b(0)} = \frac{(-1)^i (n-q-1)! q! \lambda^i \beta_i}{(q-i)! (n-q+i-1)!} = (-1)^i \frac{\binom{n-1}{q-i}}{\binom{n-1}{q}} \lambda^i \beta_i. \tag{38}$$

Then

$$\begin{aligned}
& \frac{\binom{n-1}{q}}{a(0)b(0)} \sum_{i=0}^q (-i c_\lambda^2 + (i - \mathfrak{z}) \lambda s_\lambda^2) \text{ta}_\lambda^{2i} a(i) b(i) \\
&= \sum_{i=0}^q (-i c_\lambda^2 + (i - \mathfrak{z}) \lambda s_\lambda^2) \text{ta}_\lambda^{2i} (-1)^i \binom{n-1}{q-i} \lambda^i \beta_i \\
&= -\lambda \mathfrak{z} s_\lambda^2 \binom{n-1}{q} \\
&\quad - (-s_\lambda^2 + (1 - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^2) \lambda \binom{n-1}{q-1} \beta_1 \\
&\quad + (-2s_\lambda^2 \text{ta}_\lambda^2 + (2 - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^4) \lambda^2 \binom{n-1}{q-2} \beta_2 \\
&\quad - (-3s_\lambda^2 \text{ta}_\lambda^4 + (3 - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^6) \lambda^3 \binom{n-1}{q-3} \beta_3 \\
&\quad \dots\dots\dots \\
&\quad + (-1)^q \left(-q s_\lambda^2 \text{ta}_\lambda^{2q-2} + (q - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^{2q} \right) \lambda^q \beta_q \\
&= \sum_{i=0}^{q-1} (-1)^i \lambda^{i+1} s_\lambda^2 \text{ta}_\lambda^{2i} \left((i - \mathfrak{z}) \binom{n-1}{q-i} \beta_i + (i+1) \binom{n-1}{q-i-1} \beta_{i+1} \right) \\
&\quad + (-1)^q (q - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^{2q} \lambda^q \beta_q \tag{39}
\end{aligned}$$

Multiplying numerator and denominator of (37) by $\frac{\binom{n-1}{q}}{a(0)b(0)}$, having into account (11) and (15), we obtain

$$\begin{aligned}
\mu_1(P_\rho) &\leq \mu_1(\mathfrak{P}_\rho) \\
&+ \frac{\sum_{i=0}^{q-1} (-1)^i \lambda^{i+1} \left((i - \mathfrak{z}) \binom{n-1}{q-i} \beta_i + (i+1) \binom{n-1}{q-i-1} \beta_{i+1} \right) B_i(\rho) + (-1)^q (q - \mathfrak{z}) \lambda s_\lambda^2 \text{ta}_\lambda^{2q} \lambda^q \beta_q B_q(\rho)}{\sum_{i=0}^q (-1)^i \binom{n-1}{q-i} \lambda^i \beta_i C_i(\rho)} \tag{40}
\end{aligned}$$

where $B_i(\rho) = \int_0^\rho s_\lambda^2 \text{ta}_\lambda^{2i} \nu_{\mathfrak{P}} dr$ and $C_i(\rho) = \int_0^\rho s_\lambda^{2i} c_\lambda^{-2i} \mu_{\mathfrak{P}} dr$.

When $\mathfrak{P} = \mathbb{C}P^q(\lambda)$, $\mathfrak{z} = 0$. To study the sign of $M_{\mathbb{C}P^q(\lambda)}$ we consider its expression given by the second summand in (25). In this case, from the definition (5) of h_P , one has $h_P - h_{\mathbb{C}P^q(\lambda)} = \sum_{i=1}^q \frac{k_i^2}{c_\lambda^2 (c_\lambda^2 - k_i^2 s_\lambda^2)} \geq 0$ for $r < \text{cut}(P)$. Since also $\theta \geq 0$ for $r < \text{cut}(P)$ and $f'_\mathfrak{P} < 0$ on $[0, \rho]$, one has that $M_{\mathbb{C}P^q(\lambda)} \leq 0$. On the other hand, the equality in (3) implies the equality in (23), which implies that $f_{\mathbb{C}P^q(\lambda)}$ is an eigenfunction with eigenvalue

$\mu_1(\mathbb{C}P^q(\lambda))$, which, from (24), implies $h_P - h_{\mathbb{C}P^q(\lambda)} = 0$, which, from the above expression only happen if $k_i = 0$, that is, if $P = \mathbb{C}P^q(\lambda)$.

When $\mathfrak{P} = Q^{n-1}(\lambda)$, one has $\mathfrak{z} = q = n - 1$ and there is only one polynomial defining P , with degree a_n , then $\beta_i = (a_n - 1)^i$ and the numerator of $M_{Q^{n-1}(\lambda)}$ becomes

$$\begin{aligned} & \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \left((i - (n-1)) \binom{n-1}{n-1-i} (a_n - 1)^i + (i+1) \binom{n-1}{n-1-i-1} (a_n - 1)^{i+1} \right) B_i(\rho) \\ &= \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \left((i - (n-1)) \binom{n-1}{n-1-i} + (i+1) \binom{n-1}{n-1-i-1} (a_n - 1) \right) (a_n - 1)^i B_i(\rho) \end{aligned} \quad (41)$$

But

$$\begin{aligned} (i - (n-1)) \binom{n-1}{n-1-i} &= -(n-1-i) \binom{n-1}{n-1-i} \\ &= -\frac{(n-1) \cdots (n-i-1)}{i!} = -(i+1) \binom{n-1}{n-1-i-1}, \end{aligned} \quad (42)$$

which, substituted in (41) gives

$$\begin{aligned} & \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \frac{(n-1) \cdots (n-i-1)}{i!} (-1 + a_n - 1) (a_n - 1)^i B_i(\rho) \\ &= (a_n - 2) \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \frac{(n-1) \cdots (n-i-1)}{i!} (a_n - 1)^i B_i(\rho) \end{aligned} \quad (43)$$

Then $M_{Q^{n-1}(\lambda)}$ vanishes when $a_n = 2$. As a consequence, among all the complex hypersurfaces P defined by a polynomial of degree 2, the complex hyperquadric gives the maximum value of $\mu_1(P)$.

Now, let us study the sign of $M_{Q^{n-1}(\lambda)}$. First, recall that its denominator is positive for $\rho < \text{cut}(P)$ (as we noticed in (35)). To check the sign of the numerator, the observation that when $2[(n-2)/2] + 1 > n - 2$, $n - 2[(n-2)/2] - 2 = 0$, allows us to write the sum in (43) in the following way:

$$\begin{aligned} & \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \frac{(n-1) \cdots (n-i-1)}{i!} (a_n - 1)^i B_i(\rho) \\ &= \sum_{j=0}^{[(n-2)/2]} (-1)^{2j} \lambda^{2j+1} \frac{(n-1) \cdots (n-2j-1)}{(2j)!} (a_n - 1)^{2j} \left(B_{2j}(\rho) - \lambda \frac{(n-2j-2)}{(2j+1)} (a_n - 1) B_{2j+1}(\rho) \right) \\ &= \sum_{j=0}^{[(n-2)/2]} \lambda^{2j+1} \frac{(n-1) \cdots (n-2j-1)}{(2j)!} (a_n - 1)^{2j} \int_0^\rho s_\lambda^2 \text{ta}_\lambda^{4j} \nu_{\mathfrak{P}} \left(1 - \lambda \frac{(n-2j-2)}{(2j+1)} (a_n - 1) \text{ta}_\lambda^2 \right) dr \end{aligned}$$

which is negative for $\rho \leq \rho_1 := \min_{0 \leq j \leq [(n-2)/2]} \text{ta}_\lambda^{-1} \left(\sqrt{\frac{2j+1}{\lambda(n-2j-2)}} \right)$, where ta_λ^{-1} means the inverse function of ta_λ with image in $[0, \pi/2\sqrt{\lambda}]$. Then, taking $\rho_0 = \min\{\text{cut}(P), \rho_1\}$,

we have that $M_{Q^{n-1}(\lambda)} < 0$ for $a_n \geq 3$. This gives a gap between $\mu_1(Q^{n-1}(\lambda)_\rho)$ and $\mu_1(P_\rho)$ for all complex hypersurfaces defined by polynomials of degree ≥ 3 .

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Universidad de Valencia
 Departamento de Matemáticas para la Economía y la Empresa
 Avda Tarongers s/n
 46022-Valencia (Spain)
 email: carmen.domingo@uv.es
 and
 Universidad de Valencia.
 Departamento de Geometría y Topología
 Avda. Andrés Estellés, 1, 46100-Burjassot (Valencia) Spain
 email: miquel@uv.es